The Monge equation and topology of solutions of the second order ODE's

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Abstract

Properties of the Monge equation $\Phi\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) = 0$ associated with the partial differential of equation $F_x \dot{x} + F_y \dot{y} = 0$ to the first integral $F(x, y) = C$ of the system ODE $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ are studied. Examples of solutions of the equation $P\mu_x + Q\mu_y + (P_x + Q_y)\mu = 0$ to the integrating multiplier $\mu = \mu(x, y)$ of the first order ODE $Qdx - Pdy = 0$, where $Q(x, y)$ and $P(x, y)$ are the polynomial on x, y, are constructed. Topological properties of the relations $F(x, y, a, b) = 0$ which generate dual second order ODE's $y'' = f(x, y, y')$ and $b'' = \phi(a, b, b')$ are considered.

Keywords: The Monge equation, integrating multiplier, duality, ODE.

1 Introduction

Finding integrating multiplier $\mu(x, y)$ for the equation $y' = f(x, y) = \frac{Q(x,y)}{P(x,y}$ leads to the search for solutions of linear p.d.e. $P\mu_x + Q\mu_y + (P_x + Q_y)\mu = 0$, which is a problem equivalent to solution of the equation $y' = f(x, y)$. In this paper we study general properties of the equation for the integrating multiplier using associated equations of Monge. Another problem considered here is the study of the properties of the relations $F(x, y, a, b) = 0$ which generate the second order ODE $y'' = h(x, y, y')$ and the equation $b'' = \phi(a, b, b')$ dual to it.

2 The Monge equation

Monge equation for the first order p.d.e. $H(x,y,z,z_x,z_y)=0$ or $H(x,y,z,p,q)=0$ is a result of an exclusion of the variables $p=z_x$ and $q=z_y$ from the system of equations $H(x,y,z,p,q)=0$, $\frac{dx}{H_p}=\frac{dy}{H_q}=\frac{dz}{pH_p+qH_q}$. To construct the Monge equation for the p.d.e. $P_2F_x+Q_2F_y=0$, which determines the first integral $F(x,y)=C$ of the system $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ we transform it into the form $a_0\,\omega_x+a_1\,x\omega_x+a_2\,\omega_t\omega_x+a_{11}\,x^2\omega_x+a_{12}\,x\omega_t\omega_x+a_{22}\,\omega_t^2\omega_x-b_0\,t-xb_1\,t-\omega_t b_2\,t-x^2b_{11}\,t-\omega_t x b_{12}\,t-\omega_t^2 b_{22}\,t=0$, with the help of change of the variables $F(x,y)=u(x,t)$, $y=v(x,t)$, $F_x=u_x-\frac{u_t}{v_t}v_x$, $F_y=\frac{u_t}{v_t}$, where $u(x,t)=t\omega_t-\omega$, $v(x,t)=\omega_t$, $\omega=\omega(x,t)$ and t is the parameter. To obtain Monge equation associated with this equation we restrict ourselves to the special case when the parameters are $a_0=0$, $a_1=0$, $a_2=1$, $a_{11}=0$, $a_{12}=1$, $a_{22}=0$. Remark that full quadratic system of equations is reduced to the form $(1+xy)y'=Q_2(x,y)$ by means of linear change of variables.

Theorem 1. *The Monge equation for the p.d.e. which is associated with ODE of the form* $(1+xy)dy-Q_2(x,y)dx=0$ *is*

$$\begin{gathered}x^2\,(z_x)^2+\left(4\,yb_{22}+2\,xy_x-2\,b_{12}\,x^2y-2\,b_2\,yx\right)z_x+\\+(y_x)^2+\left(-4\,x^2b_1\,y-4\,x^3b_{11}\,y+2\,b_2\,y-4\,b_0\,yx+2\,b_{12}\,xy\right)y_x+\\+x^2y^2{b_{12}}^2-4\,b_0\,y^2b_{22}-4\,xb_1\,y^2b_{22}-4\,x^2b_{11}\,y^2b_{22}+2\,xy^2b_2\,b_{12}+\\+y^2{b_2}^2=0,\end{gathered}\tag{1}$$

where $z=z(x)$ *and* $y=y(x)$.

Integral curves of the Monge equation envelope characteristic curves of the p.d.e. and so they can be useful to study properties of solutions of corresponding ODE. Properties of solutions of equation (1) as function on parameters can be studied geometrically after their consideration as quadratic integral of geodesics of 3-dimensional space with the Riemann metric $ds^2=x^2dz^2+\left(2\,dy\,x+\left(-2\,b_{12}\,x^2y-2\,b_2\,yx+4\,yb_{22}\right)dx\right)dz+dy^2+\left(-4\,x^2b_1\,y-4\,x^3b_{11}\,y+2\,b_2\,y-4\,b_0\,yx+2\,b_{12}\,xy\right)dx\,dy+{}+y^2\left(-4\,x^2b_{11}\,b_{22}+x^2{b_{12}}^2+2\,xb_2\,b_{12}-4\,xb_1\,b_{22}+{b_2}^2-4\,b_0\,b_{22}\right)dx^2$.

2.1 Integrating multiplier

The first order of p.d.e. for integrating multiplier $\mu(x,y) = \exp(h(x,y)$ for the equation $P_2dy - Q_2dx = 0$ has the form $Q_2h_y + P_2h_x + P_{2x} + Q_{2y} = 0$. To obtain the equation of Monge associated with this equation we transform it into another form with the help of Ampere transformation $x = \xi,\ y = \rho_\eta,\ z = \eta\rho_\eta - \rho(\xi,\eta),\ h_x = -\rho_\xi,\ h_y = \eta$. In result we get the equation $(\eta\, b_{22} - a_{22}\,\rho_\xi)\,(\rho_\eta)^2 + (a_{12} + 2\, b_{22} + (-a_2 - \xi\, a_{12})\,\rho_\xi + \eta\, b_2 + \eta\, b_{12}\,\xi)\,\rho_\eta +$
$+ \left(-a_{11}\,\xi^2 - a_0 - \xi\, a_1\right)\rho_\xi + b_2 + 2\, a_{11}\,\xi + b_{12}\,\xi + \eta\, b_{11}\,\xi^2 +$
$+ a_1 + \eta\, b_1\,\xi + \eta\, b_0 = 0$, from which in particular case $a_0 = 1, a_1 = 0, a_2 = 0, a_{11} = 0, a_{12} = 1, a_{22} = 0$ follows the Monge equation

$$-(z_x)^2 x^2 + \left(2\,yb_{12}\,x^2 - 2\,y_x x + 2\,yxb_2 + 2\,x - 4\,b_{22}\,y + 4\,xb_{22}\right) z_x -$$

$$-(y_x)^2 + 2\left(y(-b_{12}\,x + 2b_0\,x + 2b_1\,x^2 - b_2\right) y_x +$$

$$+2\left(-2\,b_{22} + 2\,b_{12}\,x^2 - 1 + 2\,xb_2 + 2\,yb_{11}\,x^3\right) y_x +$$

$$+4\,y^2 b_0\,b_{22} - 1 - y^2 {b_2}^2 - y^2 {b_{12}}^2 x^2 - 4\,b_{22} - 2\,y^2 b_{12}\,b_2\,x +$$

$$+4\,y^2 b_{22}\,b_{11}\,x^2 + 4\,y^2 b_{22}\,b_1\,x - 4\,{b_{22}}^2 - 2\,yb_{12}\,x - 2\,b_2\,y = 0.$$

3 On topology of General Integral

Definition. *The equations $y'' = f(x,y,y')$ and $b'' = \phi(a,b,b')$ form a dual pair if each of them is obtained by excluding variables (a,b) or (x,y) from the relation $F(x,y(x),a,b(a)) = 0$, after its double differentiation on the corresponding variable x or a* [2]. The relation $F(x,y,a,b) = 0$ in this case is a common General Integral for both equations and it can have nontrivial topological properties.

Theorem 1. *Differentiating twice the relation $F(x,y,a,b) = y(x)^2 b + x^3 + axb^2 + nb^3 = 0$ with respect to x and excluding the variables (b,a) from the obtained equalities, we get the differential equation*

$$2\,x^2\,(y(x))^3\,(y'')^3 + \left(6\,x^2\,(y')^2\,y(x)^2 + 3\,y(x)^4 - 6\,xy(x)^3 y'\right)(y'')^2 +$$

$$+\left(6\,y(x)^3(y')^2+6\,x^2(y')^4y(x)-12\,xy(x)^2(y')^3\right)y''+3\,y(x)^2(y')^4+$$
$$+27\,nx^2+2\,x^2(y')^6-6\,xy(x)(y')^5=0.$$

By analogy the dual equation is obtained

$$b(a)^6(b'')^3+\left(-9\,b(a)^5(b')^2-3\,(b(a)^4a(b')^3)\right)(b'')^2+$$
$$+\left(12\,b(a)^3(b')^5a+24\,b(a)^4(b')^4\right)b''+12\,b(a)(b')^8a^2-108\,(b')^9n^2+$$
$$+4\,a^3(b')^9-16\,b(a)^3(b')^6=0$$

Corollary 1 *The relation* $F(x,y,a,b)=y^2b+x^3+axb^2+nb^3=0$ *in variables* (x,y) *is a family of cubic curves of genus* $g=1$ *dependent on parameters* (a,b) *and arbitrary* $n=n(a)$*. But this relation, considered as family of algebraic curves in the variables* (a,b)*, can have arbitrary genus* $g=N, N=0,1,2,3...$*, for example, (*$g=0$ *if* $n=a^3$*, and* $g=2$ *when* $n=a^4$*). Appropriate classes of dual equations differ and this property can be used in the theory of* $H(x,y,y',y'')=0$*, as well as the first order theory of ODE* $h(x,y,y')=0$*.*

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