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Duality and a Riemann metrics in theory of a second order ODE's

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Abstract

A properties of integral curves of ODE's of the second order $\frac{d^2}{dx^2}y(x) = Q(x,y,y')$ and of the first order $\frac{d}{dx}y(x) = \frac{Q_n(x,y)}{P_n(x,y)}$ with a help of conformal $4D$- Riemann metrics of Fefferman and Walker types are investigated.

Keywords: CR-structures, Duality, Fefferman metrics, Bach tensor, Walker metrics).

1 Introduction

Geometric methods play an important role in theory of differential equations. Here we deal with geometry of a second order ODE $y'' = Q(x,y,y')$ considered modulo point transformations of variables. Using analogy between the $2d$- order ODE's and 3-dimensional CR-structures and theory of duality developed by E.Cartan we apply conformal $4D$-metrics of Fefferman with signature $(2,2)$ to the study of Invariant properties of integral curves of a second order ODE's having the form $y'' = A_4(x,y) + A_3(x,y)y') + A_2(x,y)y'^2 + A_1(x,y)y'^3$ and of a more complicate type. In particular properties of algebraic second order ODE's $F(y,y',y'') = 0$ and $F(x,y',y'') = 0$ which admit reduction to a first order ODE's $H(x,y,y') = 0$ are considered. With this aim Bach-tensor B_{ik} of the metric expressed through the derivatives of the Weyl tensor C_{ijkl} is applied.

2 Duality to a second order ODE's

The relation between a four variables $F(x, y, a, b) = 0$ determines some $3D$-manifold and generates two a second order ODE's $y'' = f(x, y, y')$ and $b'' = h(a, b, b')$. A first one is result of elimination of variables a, b from the conditions

$$F(x, y, a, b) = 0,\ F_x + y'F_y = 0,\ F_{xx} + 2y'F_{xy} + F_{yy}y'^2 + F_y y'' = 0,$$

and a second one is obtained after eliminating of the variables x, y from the system

$$F(x, y, a, b) = 0,\ F_a + b'F_b = 0,\ F_{aa} + 2b'F_{ab} + F_{bb}b'^2 + F_b b'' = 0.$$

Taking into account that only general integral $F(x, y, a, b) = 0$ of an arbitrary second-order equation $y'' = f(x, y, y')$ contains complete information about solutions E. Cartan developed geometric theory of Duality for various invariant classes of the second order ODE's. As example the equations of the form $y'' = A_4(x, y) + A_3(x, y)y' + A_2(x, y)y'^2 + A_1(x, y)y'^3$ and $b'' = h(a, b, b')$ where the function $h(a, b, b' = c)$ is solution of the p.d.e. $D^2h_{cc} - 4Dh_{bc} - h_c Dh_{cc} + 4h_c h_{bc} - 3h_b h_{cc} + 6h_{bb} = 0$, where $D = \partial_a + c\partial_b + h\partial_c$ form a dual pairs.

Theorem 1. *To the equation* $(b_0 + b_1y' + b_2y'^2)y'' = a_0 + a_1y' + a_2y'^2 + a_3y'^3 + a_4y'^4 + a_5y'^5$, $b = b_i(x, y), a_i = a_i(x, y)$, *from the point invariant class* $h_m(x, y, y')y'' = h_{m+3}(x, y, y'), m = 2$ *dual equation determined by solutions of the equation* $\psi_6^3 + \psi_8\psi_5^2 + \psi_4\psi_7^2 - \psi_4\psi_5\psi_8 - 2\psi_5\psi_6\psi_7 = 0$, *where* $4!\psi_4 = -\frac{d^2}{da^2}h_{cc} + 4\frac{d}{da}h_{bc} + h_c\frac{d}{da}h_{cc} - 4h_c h_{bc} + 3h_b h_{cc} - 6h_{bb}$ *and* $\psi_6^2 - \psi_4\psi_5 = 0, m = 1$, *where* $k\psi_k = \frac{d}{da}\psi_{k-1} + (3 - k)h_c\psi_{k-1} + (k - 1) + h_b\psi_{k-2}, k > 4$.

Example 1 *The equations*

$$-4 + 3x^4\, y(x)^2 + 2\, x^5 y'y'' = 0, b'' + \frac{1}{a}\tan(1/2\, b(a)) = 0$$

with General Integral

$$F(x, y, a, b) = b + 2\ \arctan(\frac{1}{\sqrt{-1 + xa}})a + 2\,\frac{\sqrt{-1 + xa}}{x} - y = 0$$

form a dual pair.

3 Fefferman metrics

Definition 1 [2]. *The Fefferman metrics is the metrics on* 4*-manifolds in coordinates* $(x, y, p = y')$ *associated with an second order ODE's* $y'' = Q(x, y, y')$*, considered modulo point transformation* $x = f(x, y), y = g(x, y)$*.* It has the form

$$ds^2 = (dz - Qdx)dx - (dy - zdx)(dw + \frac{2}{3}Q_z dx + \frac{1}{6}Q_{zz}(dy - zdx))$$

Definition 2 . *Bach tensor of* 4*-manifolds has the form*

$$B_{ik} = \nabla^r \nabla^s C_{risk} + \frac{1}{2}R^{rs}C_{irks},$$

where C_{ijkl} is Weyl tensor, R^{ij} is Ricci tensor of the metrics.

Theorem 2. *For the conformal Fefferman metrics*

$$-e^{-A(x,y,z)}ds^2 = -6\,ldzdx + 6\,lQ(x, y, z)dx^2 + dydw+$$
$$+4l\left(\frac{\partial}{\partial z}Q(x, y, z)\right)dx\,dy + l\left(\frac{\partial^2}{\partial z^2}Q(x, y, z)\right)dy^2-$$
$$-2l\left(\frac{\partial^2}{\partial z^2}Q(x, y, z)\right)zdx\,dy - zdxdw - 4l\frac{\partial}{\partial z}Q(x, y, z)\,zdx^2+$$
$$+z^2 l\frac{\partial^2}{\partial z^2}Q(x, y, z)dx^2$$

all components of Bach tensor equal to zero $B_{xx} = 0, B_{xy} = 0, B_{yy} = 0$ if the function $Q(x, y, z)$ satisfies to the p.d.e.

$$2\,QQ_{zzzz}Q_{zz}+z^2Q_{yyzzzz}+2z\,Q_{xyzzzz}+Q_{xxzzzz}+Q^2Q_{zzzzzz}+zQ_yQ_{zzzzz}+$$
$$+2z\,Q(x, y, z)Q_{yzzzzz} + 2\,Q_z^2Q_{zzzz} + 3\,Q_zQ_{xzzzz} + 2\,QQ_{xzzzz}+$$
$$+2z\,Q_{yz}Q_{zzzz} + 2\,Q_{xz}Q_{zzzz} + Q_xQ_{zzzzz} + 3z\,Q_zQ_{yzzzz}+$$
$$+4\,QQ_zQ_{zzzzz} + QQ_{yzzzz} - Q_yQ_{zzzz} = 0. \qquad (1)$$

Corollary 1 *Class of the second order ODE's* $y'' = Q(x, y, y')$ *with the function* $Q(x, y, y')$ *from [1] contains the equations of the form* $y'' = A_4(x, y) + A_3(x, y)y' + A_2(x, y)y'^2 + A_1(x, y)y'^3$ *and dual of them.*

4 Conformal Walker metrics

Definition 3 [3] . *Conformal Walker metrics on 4-manifolds in coordinates (x, y, z, w) associated with the second order ODE's $y'' = a_4(x, y)+3a_3(x, y)y'+3a_2(x, y)y'^2+a_1(x, y)y'^3$, considered modulo point transformation $x = f(x, y), y = g(x, y)$ has the form*

$$\frac{1}{2}\,e^{-A(x,y)}\,ds^2 = dx\,dz + dy\,dw + 2\,(\,za_2(x,y) -\,wa_3(x,y))\,dx\,dy+$$
$$+\,(za_3(x,y) - a_4(x,y)w)\,dx^2 + (za_1(x,y) - wa_2(x,y))\,dy^2. \qquad (2)$$

This type of metric generalizes the standard Walker metric and is applied to the study of the properties of geodesics and transfer surfaces defined by the system of equations $Z^i_{u,v} + \Gamma^i_{jk}Z^j_u Z^k_v = 0$, where Γ^i_{jk} are Christoffel symbols of the metrics.

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