Proceedings of the Fifth Conference of Mathematical Society of Moldova
IMCS-55, September 28 - October 1, 2019, Chisinau, Republic of Moldova

Riemann space in theory of the Navier-Stokes Equations

Valery Dryuma

Abstract

The Navier-Stokes system of incompressible fluids of equations is studied based on the 14-dimensional Riemann metrics. The connections between geometrical and analytical properties of the system are discussed.

Keywords: Riemann metrics, Navier-Stokes system, geodesics.

1 Introduction

A famous Navier-Stokes system of the equations

$$\vec{U}_t + (\vec{U} \cdot \vec{\nabla})\vec{U} - \mu\Delta\vec{U} + \vec{\nabla}P = 0, \quad (\vec{\nabla} \cdot \vec{U}) = 0, \tag{1}$$

which describes the flows of incompressible liquids and relates to various branches of modern mathematical physics, is a subject of intensive investigation. Numerous articles are devoted to studying of its solutions and properties. In this work we develop geometrical approach to construct new examples of solutions of given system and the relations with known results are considered.

2 14D-metrics and NS-equations

Let us introduce the 14-dim space with the metrics in the local coordinates $x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n$

$$^{14}ds^2 = 2\,dxdu + 2\,dydv + 2\,dzdw + 2\,dtdp + (-Uu - Vv - Ww)\,dt^2 +$$

$$+\Big(-u\,(U)^2 - uP - Up + u\mu\,U_x - vUV + v\mu\,U_y - wUW + w\mu\,U_z\Big)\,d\eta^2+$$

$$d\eta d\xi + \Big(u\mu\,V_x + v\mu\,V_y - Vp - wVW + w\mu\,V_z - v\,(V)^2 - vP - uUV\Big)\,d\rho^2+$$

$$+\Big(-wP - Wp + u\mu\,W_x + v\mu\,W_y + w\mu\,W_z - vVW - w\,(W)^2 - uUW\Big)\,dm^2+$$

$$+2\,dmdn + 2\,d\rho d\chi, \qquad (2)$$

where $U = U(x,y,z,t)$, $V = V(x,y,z,t)$, $W(x,y,z,t)$ and $P = P(x,y,z,t)$.

Theorema 1. *The Ricci-tensor of the metrics (2) has the components*

$$R_{44} = U_x + V_y + W_z,$$

$$R_{55} = 2\,UU_x - \mu\,U_{xx} + P_x + VU_y + UV_y - \mu\,U_{yy} + U_zW + UW_z - \mu\,U_{zz} + U_t$$

$$R_{66} = U_xV + UV_x - \mu\,V_{xx} + 2\,VV_y - \mu\,V_{yy} + P_y + V_zW + VW_z - \mu\,V_{zz} + V_t$$

$$R_{77} = U_xW + UW_x - \mu\,W_{xx} + V_yW + VW_y - \mu\,W_{yy} + 2\,WW_z - \mu\,W_{zz} + P_z + W_t$$

and it is equal to zero on solutions of the Navier-Stokes system of equations

The metrics D^{14} belongs to the class of spaces of Riemann extensions of affine connected spaces.

Namely, if we have the N-dim space in the coordinates $\vec{x}$ with the Affine Connection $\Pi^k_{ij}(\vec{x})$, then it is possible to construct the $2N$-dimensional space in the coordinates $\vec{x}, \vec{\Psi}_l$ equipped by the Riemann metrics of the form

$$^{2n}ds^2 = -2\Pi^k_{ij}\Psi_k dx^i dx^j + 2dx^l d\Psi_l.$$

Wherein full system of geodesics consists of two parts

$$\frac{d^2x^k}{ds^2} + \Pi^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \quad and \quad \frac{\delta^2\Psi_k}{ds^2} + R^l_{kij}\frac{dx^i}{ds}\frac{dx^j}{ds}\Psi_l = 0,$$

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Pi^l_{jk}\frac{dx^j}{ds}\Psi_l.$$

3 Geodesics

Part of geodesics of the metrics has the form

$$\frac{d^2}{ds^2}\xi(s) = 0, \quad \frac{d^2}{ds^2}\chi(s), \quad \frac{d^2}{ds^2}n(s) = 0,$$

$$\frac{d^2}{ds^2}\eta(s) = 0, \quad \frac{d^2}{ds^2}\rho(s), \quad \frac{d^2}{ds^2}m(s) = 0, \tag{3}$$

from where it follows

$$\xi(s) = a_1\, s, \;\; \chi(s) = a_2\, s, \;\; n(s) = a_3\, s, \;\; m(s) = a_4\, s,$$

$$\rho(s) = a_5\, s, \;\; \eta(s) = a_6\, s.$$

The remaining eight equations are of the form

$$\frac{d^2}{ds^2}x(s) - 1/2\, U \left(\frac{d}{ds}t(s)\right)^2 + 1/2\, a_6{}^2\mu\, \frac{\partial}{\partial x}U - 1/2\, a_6{}^2U^2 -$$

$$-1/2\, a_6{}^2P - 1/2\, a_5{}^2UV + 1/2\, a_5{}^2\mu\, V_x - 1/2\, a_4{}^2UW + 1/2\, a_4{}^2\mu\, W_x = 0,$$

$$\frac{d^2}{ds^2}y(s) - 1/2\, V \left(\frac{d}{ds}t(s)\right)^2 - 1/2\, a_6{}^2UV + 1/2\, a_6{}^2\mu\, U_y + 1/2\, a_5{}^2\mu\, \frac{\partial}{\partial y}V -$$

$$-1/2\, a_5{}^2V^2 - 1/2\, a_5{}^2P - 1/2\, a_4{}^2VW + 1/2\, a_4{}^2\mu\, W_y = 0,$$

$$\frac{d^2}{ds^2}z(s) - 1/2\, W \left(\frac{d}{ds}t(s)\right)^2 - 1/2\, a_6{}^2UW + 1/2\, a_6{}^2\mu\, U_z - 1/2\, a_5{}^2VW +$$

$$+1/2\, a_5{}^2\mu\, V_z + 1/2\, a_4{}^2\mu\, W_z - 1/2\, a_4{}^2W^2 - 1/2\, a_4{}^2P = 0,$$

$$\frac{d^2}{ds^2}t(s) - 1/2\, U\, a_6{}^2 - 1/2\, V\, a_5{}^2 - 1/2\, W\, a_4{}^2 = 0,$$

and the linear system of the second order ODE's to the coordinates $u(s), \; v(s), \; w(s), \; p(s)$.

Theorema 2. *Elimination of the variables u, v, w, p from the conditions to the non zero components of the Ricci-tensor*

$$R_{ij} - \lambda\, g_{ij} = 0,$$

lead to the condition on the

$$-\mu\, U_yUVP+\mu^2W_yU_xVW-\mu^2UVW_yU_z-P^2V^2-U^2P^2+\mu^2UWV_yU_z-$$
$$-\mu\, W_yPVW-\mu^2V_xW_yUW-\mu\, W_xPUW+\mu^2W_xV_yUW-\mu^2U_yW_xVW-$$
$$-P^2W^2+\mu\, PV_yW^2-\mu\, V_xUVP+\mu^2V_xU_yW^2+\mu^2W_xV^2U_z+$$
$$+\mu\, U_xPW^2-\mu^2U_xV_yW^2-\mu\, UWPU_z-\mu^2V_xVWU_z-\mu^2U_yUW_z-$$
$$-\mu^2UVW_xV_z-\mu\, VWPV_z+\mu^2W_yU^2V_z+\mu^2VWU_xV_z+\mu^2U^2V_y^2+$$
$$+\mu^2U_x^2V^2-\mu^2V_xUVU_x+V_y\mu^2U_xV^2-V_y\mu\, PV^2-U_x\mu\, U^2P+$$
$$+U_x\mu^2U^2V_y-\mu^2V_xUVV_y-\mu^2U_yUVU_x-\mu^2U_yUVV_y=0,$$

which can be used to the theory of the NS-equations.

4 Conclusion

As it was shown, the system of the Navier-Stokes equations, which describes a flow of incompressible liquids, admits geometric consideration on the basis of Riemann metrics of 14-dim space. In such a way, new possibilities for investigation of its properties arise.

Acknowledgments. The article has been supported by SCSTD of ASM (project No. 15-817-023.03F).

References

[1] V. Dryuma, *The Ricci flat space related with the Navier-Stokes equations*. Buletinul Academiei de Stiintse a Republicii Moldova, Matematica, no. 2(69), 2012, p.99–102.

Valery Dryuma

Vladimir Andrunachievichi Institute of Mathematics and Computer Science
E–mail: valdryum@gmail.com