RIEMANN AND PROJECTIVE SPACES IN THEORY OF THE FIRST AND SECOND ORDER ODE's

Valerii Dryuma

Moldova State University, Chişinău, Republic of Moldova

valdryum@gmail.com

The second order ODE's of the form

$$y_{xx} + a_1(x,y)y_x^3 + 3\,a_2(x,y)y_x^2 + 3\,a_3(x,y)y_x + a_4(x,y) = 0, \qquad (1)$$

where the coefficients $a_i(x,y)$ are arbitrary smooth functions, conserve their form under non degenerated changes of variables $x = x(u,v), \quad y = y(u,v)$. We shall use this type of equations to study the properties of solutions of polynomial systems $dy_s = Q_n(x(s), y(s), b_i), dx_s = P_n(x(s), y(s), b_i)$ with parameters b_i.

Theorem 1. [1] *The equations of geodesical lines of four-dimensional Riemann metric with coordinates* $[x, y, z, t]$

$$ds^2 = (za_3(x,y) - ta_4(x,y))\,dx^2 + dxdz + dydt+$$

$$(za_2(x,y) - ta_3(x,y))\,dxdy + (za_1(x,y) - ta_3(x,y))\,dy^2$$

for coordinates $x(s), y(s)$ *have the form*

$$y_{ss} + a_4x_s^2 + 2\,a_3x_sy_s + a_2y_s^2 = 0, \; x_{ss} - a_3x_s^2 - 2\,a_2x_sy_s - a_1y_s^2 = 0,$$

which are equivalent to the equation (1) and they are applied then to construct its solutions [1].

To study the projective properties of the equation (1) the following results are applied:

Theorem 2. [2] *The quotient space of complex projective planes is a* $4D$ *sphere, which is described by the intersection of two algebraic equations of the form*

$$a^2x - 2\,abc + b^2y + c^2z - xyz = 0, \quad x + y + z - 1 = 0.$$

As result the complex projective space $CP(2)$ in the coordinates: $z_1 = x + I \times a$, $z_2 = y + I \times b$, $z_3 = z + I \times c$ is obtained, and in this way an algebraic 3-dim hypersurface in the standard $4D$-sphere arises.

Theorem 3. [3] *The system of equations*

$$\Phi_{xy} = A^{1}_{12}\Phi_x + A^{2}_{12}\Phi_y + A_{12}\,\Phi, \quad \Phi_{xz} = A^{1}_{13}\Phi_x + A^{3}_{13}\Phi_z + A_{13}\,\Phi,$$

$$\Phi_{yz} = A^{2}_{23}\Phi_y + A^{3}_{22}\Phi_y + A_{23}\Phi$$

with coefficients $A^i_{jk} = A^i_{jk}(x, y, z)$ *defines a triple conjugate system of lines in a* 4*-dimensional projective space, whose invariants coincide with Liouville-Tresse-Cartan invariants of equation (1).*

References:

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