Proceedings of the International Conference IMCS-60, MSU
October 10–13, 2024, Chisinau, Republic of Moldova

The Riemann Spaces in the Theory of the Navier-Stokes Equations

Valerii Dryuma

Abstract

A geometric approach for integrating the system of Navier-Stokes equations in the Euler and Lagrange variables is proposed.

Keywords: space of the Riemann, tensor Ricci, 6D-metric, Invariants, geodesics, incompressibility.

1 Introduction

The basis of the geometric approach for studying properties of Navier-Stokes system of equations (NS-system) composes the relations $x = f_1\left(a,b,c,t\right),\ y = f_2\left(a,b,c,t\right),\ z = f_3\left(a,b,c,t\right),\ p = h\left(a,b,c,t\right)$, where (x,y,z,p) are the local coordinates for variable of pressure p and the variables (a,b,c) – for dual coordinates of individual points of the liquid at each moment of the parameter time t. As a result of elimination of the variables (a,b,c) with the help of the appropriate differentiations, the well-known system of Navier-Stokes equations in Euler variables is obtained

$$\frac{\partial}{\partial\eta}\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} + -\frac{1}{\rho}\vec{\nabla}P - \mu\Delta\vec{U} = 0, \quad div\ \vec{U} = 0, \tag{1}$$

where $\vec{U}$ is the vector of velocity of fluid, ρ is density, and μ is viscosity of fluid.

Similarly, after excluding of the coordinates (x,y,z) from the indicated relations, a system of equations, determining the movement of individual points of the liquid is obtained.

2 Ricci-flat metric associated with the NS-system

The $14D$-space in local coordinates $(x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n)$ is considered.

Theorem 1. *The metric of the form*

$$\begin{gathered} ds^2 = 2\,dx^2 + 2\,dxdy + 2\,dxdu + 2\,dy^2 + 2\,dydz + 2\,dydv + \\ 2\,dz^2 + 2\,dzdw + 2\,dtdp + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn + A dt^2 + \\ B d\eta^2 + C d\rho^2 + E dm^2, \end{gathered} \tag{2}$$

where

$$A = 2 - U(x, y, z, t)\,u - V(x, y, z, t)\,v - W(x, y, z, t)\,w,$$

$$\begin{gathered} B = \left(-UW + \mu \frac{\partial}{\partial z} U\right) w + \left(-UV + \mu \frac{\partial}{\partial y} U\right) v + \\ \left(\mu \frac{\partial}{\partial x} U - (U)^2 - P\right) u - Up, \end{gathered} \tag{3}$$

$$\begin{gathered} C = \left(-VW + \mu \frac{\partial}{\partial z} V\right) w + \left(\mu \frac{\partial}{\partial y} V - (V)^2 - P\right) v + \\ \left(-UV + \mu \frac{\partial}{\partial x} V\right) u - Vp, \end{gathered} \tag{4}$$

$$\begin{gathered} E = \left(-\mu \frac{\partial}{\partial x} U - \mu \frac{\partial}{\partial y} V - (W)^2 - P\right) w + \left(-VW + \mu \frac{\partial}{\partial y} W\right) v + \\ \left(-UW + \mu \frac{\partial}{\partial x} W\right) u - Wp, \end{gathered} \tag{5}$$

is the Ricci-flat on solutions of Navier-Stokes system of equations.

In the Lagrange variables (a, b, c, t), full system of Navier-Stokes equations looks moore complicated and has the form

$$\begin{gathered} \frac{\partial^2 A}{\partial t^2} + [B, C, P)] - \\ \mu\left([B, C, [B, C, A_t]] + [C, A, [C, A, A_t]] + [A, B, [A, B, A_t]]\right) = 0, \end{gathered} \tag{6}$$

$$\begin{gathered}\frac{\partial^2 B}{\partial t^2}+[C,A,P]-\\ \mu\left([B,C,[B,C,B_t]]+[B,A,[C,A,B_t]]+[A,B,[A,B,B_t]]\right)=0,\end{gathered}\tag{7}$$

$$\begin{gathered}\frac{\partial^2 C}{\partial t^2}+[A,B,P]-\\ \mu\left([B,C,[B,C,C_t]]+[C,A,[C,A,C_t]]+[A,B,[A,B,C_t]]\right)=0,\\ [A,B,C]-1=0.\end{gathered}\tag{8}$$

3 6D-metric

Theorem 2. *The* $6D$*-Riemann space in coordinates* (x,y,z,a,b,c,t)*, equipped with a metric of the form*

$$\begin{gathered}ds^2=A\left(a,b,c,t\right)dx^2+2\,B\left(a,b,c,t\right)dxdy+2\,E\left(a,b,c,t\right)dxdz+\\ dxda+C\left(a,b,c,t\right)dy^2+2\,H\left(a,b,c,t\right)dydz+dydb+\\ F\left(a,b,c,t\right)dz^2+dzdc\end{gathered}\tag{9}$$

can be used for integration of the equation

$$\begin{gathered}\frac{\partial}{\partial b}B\frac{\partial}{\partial c}C\frac{\partial}{\partial a}A-\frac{\partial}{\partial b}B\frac{\partial}{\partial a}C\frac{\partial}{\partial c}A-\frac{\partial}{\partial c}C\frac{\partial}{\partial a}B\frac{\partial}{\partial b}A-\frac{\partial}{\partial b}C\frac{\partial}{\partial c}B\frac{\partial}{\partial a}A+\\ \frac{\partial}{\partial b}C\frac{\partial}{\partial a}B\frac{\partial}{\partial c}A+\frac{\partial}{\partial c}B\frac{\partial}{\partial a}C\frac{\partial}{\partial b}A-1=0,\end{gathered}\tag{10}$$

which is the condition of incompressibility of liquid.

4 Projective duality to the theory of the second order ODE's

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y\left(x\right)+a_1\left(\frac{\mathrm{d}}{\mathrm{d}x}y\left(x\right)\right)^3+3\,a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}y\left(x\right)\right)^2+3\,a_3\,\frac{\mathrm{d}}{\mathrm{d}x}y\left(x\right)+a_4=0,\tag{11}$$

where $a_i=a_i(x,y)$.

The general integral of this equation has the form $H(x, y, a, b) = 0$ among four variables $(x, y, a, b) = 0$, from which follows a differential equation with respect to the variables a, b

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2} b\left(a\right) = g\left(a, b, \frac{\mathrm{d}}{\mathrm{d}a} b\left(a\right)\right). \tag{12}$$

Theorem 3. *Two equations* $y'' = -a_1(x, y)y'^3 - 3a_2(x, y)y'^2 - 3a_3(x, y)y' - a_4(x, y)$ *and* $b" = g(a, b, b')$*, where* $b' = c$ *form a dual pair of equations with a common integral of the form* $H(x, y, a, b) = 0$*, which with the help of solutions of the equation*

$$h\left(a,b,c\right) = \frac{\partial^2}{\partial c \partial a} g\left(a,b,c\right) + g\left(a,b,c\right) \frac{\partial^2}{\partial c^2} g\left(a,b,c\right) - 1/2 \left(\frac{\partial}{\partial c} g\left(a,b,c\right)\right)^2$$
$$+ c \frac{\partial^2}{\partial c \partial b} g\left(a,b,c\right) - 2 \frac{\partial}{\partial b} g\left(a,b,c\right)$$
$$\frac{\partial^2}{\partial c \partial a} h\left(a,b,c\right) + g\left(a,b,c\right) \frac{\partial^2}{\partial c^2} h\left(a,b,c\right) - \left(\frac{\partial}{\partial c} g\left(a,b,c\right)\right) \frac{\partial}{\partial c} h\left(a,b,c\right) +$$
$$c \frac{\partial^2}{\partial c \partial b} h\left(a,b,c\right) - 3 \frac{\partial}{\partial b} h\left(a,b,c\right) = 0, \tag{13}$$

is determined.

5 4D-metric

The geometric method for studying the properties of the NS-system of equations with the theory of second-order ODE's of normal projective connection of E. Cartan is associated.

Theorem 4. *Geodesic equations of the* $4D$*-space with the metric*

$$ds^2 = \left(2\,z a_3(x,y) - 2\,t a_4(x,y)\right) dx^2 + 2\,\left(2\,z a_2(x,y) - 2\,t a_3(x,y)\right) dx dy +$$
$$2\,dx dz + \left(2\,z a_1(x,y) - 2\,t a_2(x,y)\right) dy^2 + 2\,dy dt, \tag{14}$$

in the coordinates (x, y, z, t) decomposes into two parts.

The first part consists of two equations:

$$\frac{d^2}{ds^2}y(s) + a_4(x,y)\left(\frac{d}{ds}x(s)\right)^2 + 2\,a_3(x,y)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) + a_2(x,y)\left(\frac{d}{ds}y(s)\right)^2 = 0, \tag{15}$$

$$\frac{d^2}{ds^2}x(s) - a_3(x,y)\left(\frac{d}{ds}x(s)\right)^2 - 2\,a_2(x,y)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) - a_1(x,y)\left(\frac{d}{ds}y(s)\right)^2 = 0. \tag{16}$$

And the second part has the form of the linear system of equations for the vector $\vec{\Psi} = (\Psi_1 = z(s), \Psi_2 = t(s)$

$$\frac{d^2\vec{\Psi}}{ds^2} + A(x,y)\frac{d\vec{\Psi}}{ds} + B(x,y)\vec{\Psi} = 0, \tag{17}$$

where $\vec{\Psi} = (\Psi_1 = z(s),\ \Psi_2 = t(s))$ and $A(x,y), B(x,y)$ are the 2×2 matrix-functions.

Corresponding Ricci tensor R_{ij} of the metrics is of the form

$$\begin{aligned}
R_{11} &= 2\,\frac{\partial}{\partial y}a_4(x,y) - 2\,\frac{\partial}{\partial x}a_3(x,y) - 4\,a_3(x,y)^2 + 4\,a_4(x,y)a_2(x,y),\\
R_{12} &= -2\,\frac{\partial}{\partial x}a_2(x,y) + 2\,\frac{\partial}{\partial y}a_3(x,y) - 2\,a_2(x,y)a_3(x,y) + 2\,a_4(x,y)a_1(x,y),\\
R_{22} &= -2\,\frac{\partial}{\partial x}a_1(x,y) + 2\,\frac{\partial}{\partial y}a_2(x,y) + 4\,a_1(x,y)a_3(x,y) - 4\,a_2(x,y)^2.
\end{aligned} \tag{18}$$

6 Conclusion

This article proposes a geometric approach for constructing exact solutions of the Navier-Stokes equations and studying their properties.

Acknowledgments. The author was supported partially by the project $N.817.02.3F$ and partially by the Institutional Research Program 011303 SATGED for 2024-2027, Moldova State University.

References

[1] V. S. Dryuma. *On spaces related to the Navier-Stokes equations*, Bul. Acad. Ştiinţe Repub. Moldova, Mat., vol. 3(64), 2010, pp. 107–110.

[2] V. S. Dryuma. *The Ricci-flat spaces related to the Navier-Stokes equations*, Bul. Acad. Ştiinţe Repub. Moldova, Mat., vol. 2(69), 2012, pp. 99–102.

[3] V. S. Dryuma. *Projective duality in theory of second order differential equations*, Mathematical Research, vol. 112, Chisinau, "Stiinta", 1990, pp. 93–103. (in Russian).

[4] V. S. Dryuma. *The Riemann and Einsten-Weyl geometries in theory of differential equations, their applications and all that*, A. B. Shabat et al. (eds.), New Trends in Integrability and Partial Solvability, Kluwer Academic Publishers, Printed in the Netherlands, 2004, pp. 115–156.

Valerii Dryuma

Moldova State University, Institute of Mathematics and Computer Science "Vladimir Andrunachievici", Chisinau, Republic of Moldova
E–mail: valdryum@gmail.com
ORCID: https://orcid.org/0009-0002-6581-6650